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C^*-EXTREME POINTS

BY

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ABSTRACT. Let $\mathcal{Q}$ be a C^*-algebra and let $\mathbb{S}$ be a subset of $\mathcal{Q}$. $\mathbb{S}$ is C^*-convex if whenever $T_1, T_2, \ldots, T_n$ are in $\mathbb{S}$ and $A_1, \ldots, A_n$ are in $\mathcal{Q}$ with $\sum_{i=1}^n A_i^* A_i = I$, then $\sum_{i=1}^n A_i^* T_i A_i$ is in $\mathbb{S}$. An element T in $\mathbb{S}$ is called C^*-extreme in $\mathbb{S}$ if whenever $T = \sum_{i=1}^n A_i^* T_i A_i$ with T_i and A_i as above and with A_i invertible, then T_i is unitarily equivalent to T for each i. We investigate the linear extreme points and C^*-extreme points for three sets: first, the unit ball of operators in Hilbert space; next, the set of 2×2 matrices with numerical radius bounded by 1; and last, the unit interval of positive operators on Hilbert space. In particular we find that for the second set, the linear and C^*-extreme points are different.

The numerical range of an operator T may be defined as the set of all complex numbers λ for which there is a state ϕ on the C^*-algebra generated by T with $\phi(T) = \lambda$. Using the natural generalization of state, viz. the unital completely positive maps, one can define a "generalized" numerical range as follows: fix a Hilbert space $\mathcal{H}$ of dimension n (possibly infinite) and let

$$W_n(T) = \{\phi(T) | \phi \text{ is a unital completely positive map from } C^*(T) \text{ into } \mathcal{B}(\mathcal{H})\}.$$

The sequence of sets $W_n(T)$ contain a great deal of information about the original operator; for example, if T is compact and irreducible then the $W_n(T)$ with n finite form a complete set of unitary invariants for T [1]. Each of these sets satisfies a very strong convexity property: if $S_i \in W_n(T)$ and if $A_1, \ldots, A_k$ are operators in $\mathcal{H}$ satisfying $A_1^* A_1 + \cdots + A_k^* A_k = I$, then $A_1^* S_1 A_1 + \cdots + A_k^* S_k A_k \in W_n(T)$.

This concept, C^*-convexity, and a notion of C^*-extreme point are studied in [4]. However, it is not determined there whether C^*-extremity is distinct from linear extremity. The main point of this article is to give an example in which the two concepts are distinct: the C^*-extreme points form a proper subset of the linear extreme points of the set of all 2×2 matrices whose numerical radius is at most 1. In the process, we obtain geometrical and algebraic characterizations of these sets (Theorem 2.9 and Theorem 2.7 combined with Corollary 2.5).

§0 consists of a few preliminary definitions and results. In §1 we show that for the unit ball in $\mathcal{B}(\mathcal{H})$, the set of C^*-extreme points coincides with the set of linear extreme points. (This settles in the affirmative a conjecture in [4].) If a precompact subset S of a C^*-algebra has closed (linearly) convex hull $h(S)$, then the extreme

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points of $h(S)$ are contained in $\bar{S}$ [2, p. 440]. This is not true for C^*-extremity. We show in §3 that the set $\{T \in \mathfrak{B}(\mathcal{H}) | 0 \leq T \leq I\}$ is generated as a closed C^*-convex set by a single extreme point: any projection with infinite rank and corank. We also identify the sets singly generated by a projection with finite rank or with finite corank.

0. Definitions and preliminaries. If $\mathcal{Q}$ is a C^*-algebra and $\{T_i\}_{i=1}^n$ is a finite collection of elements in $\mathcal{Q}$, and if $\{A_i\}_{i=1}^n$ is a collection of elements such that $\sum_{i=1}^n A_i^* A_i = I$, we refer to the sum $\sum_{i=1}^n A_i^* T_i A_i$ as a C^*-convex combination of the elements $T_1, \ldots, T_n$, with coefficients $A_1, \ldots, A_n$. If $\mathcal{S}$ is a subset of $\mathcal{Q}$ with the property that every C^*-convex combination of elements from $\mathcal{S}$ must again lie in $\mathcal{S}$, we say that $\mathcal{S}$ is C^*-convex. Note that in this definition we do not restrict the coefficients $A_1, \ldots, A_n$ to lie in $\mathcal{S}$, only the elements $T_1, \ldots, T_n$. Our interest is in the case when $\mathcal{Q}$ is the set of bounded operators on a Hilbert space $\mathcal{H}$; henceforth, we assume this. We repeat here two remarks from [4].

(1) If $\mathcal{S}$ is C^*-convex, then it is convex in the usual sense. (We shall refer to the usual notion of convexity as linear convexity in order to distinguish the two concepts.)

(2) If $\mathcal{S}$ is C^*-convex then it is closed under unitary equivalence.

In accordance with [4] we denote by MCL(T) the intersection of all norm-closed C^*-convex sets containing the operator T; it is clear that MCL(T) is closed and C^*-convex.

If the coefficients $A_1, \ldots, A_n$ are all invertible we call $\sum_{i=1}^n A_i^* T_i A_i$ a proper C^*-convex combination of $T_1, \ldots, T_n$. If T is any operator and U is unitary, let $W = U^* T U$ and observe that

$$T = \left(\frac{U}{\sqrt{2}}\right) W \left(\frac{U}{\sqrt{2}}\right)^* + \left(\frac{U}{\sqrt{2}}\right) W \left(\frac{U}{\sqrt{2}}\right)^*;$$

that is, T can always be written as a proper C^*-convex combination of elements of MCL(T). Thus the definition of C^*-extreme point cannot look exactly like the definition of linear extreme point. In [4], an element T in a C^*-convex set $\mathcal{S}$ is said to be a C^*-extreme point of $\mathcal{S}$ if, whenever T is a proper C^*-convex combination of $T_1, \ldots, T_n$, then each T_i is unitarily equivalent to T. Observe that the presence of the unitary equivalence clause means that a C^*-extreme point of a set is not automatically linearly extreme.

We will require the following facts from [4]:

THEOREM 0.1. *Let $\mathfrak{B}(\mathcal{H})$ be the algebra of bounded operators on a Hilbert space $\mathcal{H}$. The following sets are C^*-convex*:

(i) $B_1 = \{T \in \mathfrak{B}(\mathcal{H}): \|T\| \leq 1\}$.

(ii) $W_1 = \{T \in \mathfrak{B}(\mathcal{H}): |(Tf, f)| \leq 1$ *for all f of norm* $1\}$.

(iii) $\mathcal{P} = \{T \in \mathfrak{B}(\mathcal{H}): T$ *is Hermitian and* $0 \leq T \leq I\}$.

THEOREM 0.2. *Let $\mathcal{S}$ be a C^*-convex subset of $\mathfrak{B}(\mathcal{H})$, and let $T \in \mathcal{S}$. The following statements are equivalent*:

(i) *T is a C^*-extreme point of $\mathcal{S}$*;

(ii) *whenever* P_1, P_2, T_1, T_2 *are operators such that* P_1 *and* P_2 *are invertible and positive,* $P_1^2 + P_2^2 = I$, $T_1, T_2 \in \mathcal{S}$, *and* $T = P_1T_1P_1 + P_2T_2P_2$, *then* T_1 *and* T_2 *are unitarily equivalent to* T.

THEOREM 0.3. *If* $\mathcal{H}$ *is finite dimentional and* $\mathcal{S}$ *is a* C^**-convex subset of* $\mathcal{B}(\mathcal{H})$, *then every* C^**-extreme point of* $\mathcal{S}$ *is linearly extreme.*

1. Unit ball. Let $\mathcal{H}$ be a Hilbert space and denote by $\mathcal{B}(\mathcal{H})$ the collection of bounded linear operators on $\mathcal{H}$, and by B the unit ball of $\mathcal{B}(\mathcal{H})$; that is, $B = \{T \in \mathcal{B}(\mathcal{H}): \|T\| \leq 1\}$. In [4] it is shown that B is C^*-convex and that each unitary operator is a C^*-extreme point of B. In this section we modify slightly the technique of [4] to show that the C^*-extreme points of B are exactly the isometries and co-isometries.

THEOREM 1.1. *Let* $U \in \mathcal{B}(\mathcal{H})$ *be either an isometry or a co-isometry. Then* U *is a* C^**-extreme point of the unit ball in* $\mathcal{B}(\mathcal{H})$.

PROOF. By Theorem 0.2 we assume that U is an isometry and that $U = P_1T_1P_1 + P_2T_2P_2$, where P_1 and P_2 are positive and invertible and $P_1^2 + P_2^2 = I$, and $T_1, T_2 \in B$. Our task is to show that T_1 and T_2 are unitarily equivalent to U. Consider the following equation involving operators on $\mathcal{H} \oplus \mathcal{H}$:

$$\begin{pmatrix} U^* & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} P_1 & P_2 \\ -P_2 & P_1 \end{pmatrix}\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}\begin{pmatrix} P_1 & -P_2 \\ P_2 & P_1 \end{pmatrix}$$
$$= \begin{pmatrix} I & U^*(-P_1T_1P_2 + P_2T_2P_1) \\ -P_2T_1P_1 + P_1T_2P_2 & P_2T_1P_2 + P_1T_2P_1 \end{pmatrix}.$$

The second and fourth matrices on the left-hand side are unitary and the norms of the first and third matrices are no greater than one. Hence, the norm of the right-hand side must be no greater than one. It follows that the off-diagonal entries must be 0:

(a) $-P_2T_1P_1 + P_1T_2P_2 = 0$,

(b) $U^*(-P_1T_1P_2 + P_2T_2P_1) = 0$.

Because P_1 and P_2 commute we can write $T_2 = (P_1^{-1}P_2)T_1(P_1P_2^{-1})$, using (a). Then

$$\begin{aligned} U = P_1T_1P_1 + P_2T_2P_2 &= P_1T_1P_1 + P_2P_1^{-1}P_2T_1P_1P_2^{-1}P_2 \\ &= P_1T_1P_1 + P_2^2P_1^{-1}T_1P_1 \\ &= P_1T_1P_1 + (I - P_1^2)P_1^{-1}T_1P_1 = P_1^{-1}T_1P_1. \end{aligned}$$

Thus, T_1 is similar to U: $T_1 = P_1UP_1^{-1}$. Similarly, $T_2 = P_2UP_2^{-1}$. Now equation (b) becomes

$$U^*(-P_1^2UP_1^{-1}P_2 + P_2^2UP_2^{-1}P_1) = 0,$$

or

$$U^*(-UP_1^{-1}P_2 + P_2^2U(P_1^{-1}P_2 + P_2^{-1}P_1)) = 0,$$

and since $U^*U = I$ and $P_1^2 + P_2^2 = I$, we have

$$-P_1^{-1}P_2 + U^*P_2^2U(P_1^{-1}P_2^{-1}) = 0.$$

Multiplying the last equation by P_1P_2 on the right, we obtain $U^*P_2^2U = P_2^2$. Similarly, $U^*P_1^2U = P_1^2$. Next,

$$T_1^*T_1 = (P_1UP_1^{-1})^*(P_1UP_1^{-1}) = P_1^{-1}U^*P_1^2UP_1^{-1}$$
$$= P_1^{-1}P_1^2P_1^{-1} = I$$

and thus T_1 and T_2 are isometries. Since T_1 and U are similar, we know that $\dim\ker T_1^* = \dim\ker U^*$. By the Wold decomposition, T_1 and U are each the direct sum of a unitary part and some copies of the unilateral shift; since the kernels of T_1^* and U^* have the same dimension, the number of copies of the shift is the same for T_1 and for U. Thus T_1 and U are unitarily equivalent if and only if the unitary parts are equivalent.

Let $\mathfrak{M}$ and $\mathfrak{N}$ be the subspaces of $\mathcal{H}$ on which T_1 and U are unitary, respectively. By [3, Problem 107],

$$\mathfrak{M} = \bigcap_{n=0}^{\infty} T_1^n\mathcal{H} \quad \text{and} \quad \mathfrak{N} = \bigcap_{n=0}^{\infty} U^n\mathcal{H}.$$

Now

$$P_1\mathfrak{N} = \bigcap_n P_1U^n\mathcal{H} = \bigcap_n P_1U^nP_1^{-1}\mathcal{H} = \bigcap_n T_1^n\mathcal{H} = \mathfrak{M}.$$

Hence, P_1 is an invertible operator that maps $\mathfrak{N}$ onto $\mathfrak{M}$. If $f \in \mathfrak{N}$

$$P_1Uf = T_1P_1f$$

and it follows that $P_1U|_{\mathfrak{N}} = (T_1|_{\mathfrak{M}})P_1$, that is, $U|_{\mathfrak{N}}$ and $T_1|_{\mathfrak{M}}$ are similar unitary operators, and thus unitarily equivalent [3, Problem 152]. Hence, also T_1 and U are unitarily equivalent, and of course the same is true for T_2 and U. The proof is complete. ■

COROLLARY 1.2. *The C^*-extreme points of the unit ball are precisely the isometries and co-isometries.*

PROOF. Suppose that T is not an isometry or co-isometry and $\|T\| < 1$. Then by [3, Problem 107], there exist T_1 and T_2, each of which is either an isometry or co-isometry, such that $T = \frac{1}{2}(T_1 + T_2)$. T cannot be unitarily equivalent to T_1 and T_2, and we are done. ■

2. Unit numerical range. In this section we will discuss the C^*-extreme points of the set of 2×2 matrices whose numerical radius is bounded by 1. In the process we will identify all the linear extreme points of this set. This example is of particular importance because some of the linear extreme points are not C^*-extreme.

Throughout this section we operate on a two-dimensional complex Hilbert space $\mathcal{H}$, so that operators are represented as 2×2 matrices. The numerical range of an operator T, denoted by $W(T)$, is the collection of complex numbers (Tf, f), where f runs through all vectors of norm 1; equivalently (for finite-dimensional spaces), the numerical range is the set of images of T under all positive unital maps from $\mathcal{B}(\mathcal{H})$ into the complex numbers. The numerical radius of T, $w(T)$, is defined by

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\}.$$

We denote by W_1 the set of matrices T such that $w(T) \leq 1$. It is a standard fact that W_1 is linearly convex, and in [4] it is shown that W_1 is C^*-convex.

LEMMA 2.1. *Let T be a 2×2 matrix with distinct eigenvalues α and β, and corresponding unit eigenvectors f and g. The numerical range of T is an elliptical disk with foci at α and β. If $\gamma = |(f, g)|$ and $\delta = \sqrt{1-\gamma^2}$ then the minor axis is $\gamma|\alpha - \beta|/\delta$ and the major axis is $|\alpha - \beta|/\delta$. If T has only one eigenvalue α, the numerical range is the circular disk of radius $\frac{1}{2}\|T - \alpha I\|$ and center α.*

The statement of Lemma 2.1 appears in [3, p. 109]. The next lemma also appears in [3, p. 324]; we prove it here for completeness.

LEMMA 2.2. *For any operator T, $w(T) \leq 1$ if and only if $\operatorname{Re}(I - zT)$ is positive for all complex numbers z of absolute value 1. Moreover, if $\|f\| = |z| = 1$ and if $w(T) \leq 1$, then $(Tf, f) = z$ if and only if $\operatorname{Re}(I - \bar{z}T)f = 0$.*

PROOF. For any operator S and vector f, $((\operatorname{Re} S)f, f) = \operatorname{Re}(Sf, f)$. The first statement of the lemma follows because for any complex number λ, $|\lambda| \leq 1$ if and only if $\operatorname{Re}(1 - z\lambda) \geq 0$ for all z of absolute value 1. Now suppose that $w(T) \leq 1$ and that $\|f\| = |z| = 1$. Then $(\operatorname{Re}(I - \bar{z}T)f, f) = \operatorname{Re}((I - \bar{z}T)f, f) = 1 - \operatorname{Re}[\bar{z}(Tf, f)]$. If $\operatorname{Re}(I - \bar{z}T)f = 0$ then $\operatorname{Re} \bar{z}(Tf, f) = 1$ and since $|\bar{z}| = 1$ and $|(Tf, f)| \leq 1$ it must be that $(Tf, f) = z$. On the other hand, if $(Tf, f) = z$ then $(\operatorname{Re}(I - \bar{z}T)f, f) = 0$; but $\operatorname{Re}(I - \bar{z}T)$ is a positive operator and thus $\operatorname{Re}(I - \bar{z}T)f = 0$. ■

We shall denote by W_1^1 the collection of matrices T for which $w(T) = 1$ and $1 \in W(T)$. W_1^1 is far from being convex (consider $\frac{1}{2}P + \frac{1}{2}(I - P)$, where P is a projection of rank one), but there is a notion of "extremity" in W_1^1 which will prove useful; Corollary 2.5 will show why.

LEMMA 2.3. *Suppose that T is a C^*-convex combination of T_1 and T_2. Then $W(T)$ is contained in the (linear) convex hull of $W(T_1) \cup W(T_2)$.*

PROOF. Suppose that $T = P_1T_1P_1 + P_2T_2P_2$, where P_1 and P_2 are positive operators and $P_1^2 + P_2^2 = I$. Let f be a unit vector; note that $\|P_1f\|^2 + \|P_2f\|^2 = \|f\|^2 = 1$. If both P_1f and P_2f are nonzero, then $f_i = P_if/\|P_if\|$ is a unit vector for $i = 1, 2$. We have

$$\begin{aligned}(Tf, f) &= (P_1T_1P_1f, f) + (P_2T_2P_2f, f)\\ &= (T_1P_1f, P_1f) + (T_2P_2f, P_2f)\\ &= \|P_1f\|^2(T_1f_1, f_1) + \|P_2f\|^2(T_2f_2, f_2).\end{aligned}$$

Thus (Tf, f) is a linear convex combination of (T_1f_1, f_1) and (T_2f_2, f_2). On the other hand, if one of the P_if is zero, say $P_1f = 0$, then

$$(Tf, f) = (P_2T_2P_2f, f) = \|P_2f\|^2(T_2f_2, f_2) = (T_2f_2, f_2). \quad ■$$

COROLLARY 2.4. *Let T, T_1, and T_2 be as above and suppose that $T, T_1, T_2 \in W_1$. If λ is a complex number in $W(T)$ such that $|\lambda| = 1$, then $\lambda \in W(T_1)$ and $\lambda \in W(T_2)$.*

PROOF. λ is a convex combination of two numbers λ_1 and λ_2 in $W(T_1)$ and $W(T_2)$ respectively. Since $|\lambda_1|, |\lambda_2| \leq 1$ and since λ is an extreme point of the unit ball in the complex plane, it follows that $\lambda_1 = \lambda_2 = \lambda$. ■

COROLLARY 2.5. *Let* $w(T) = 1$. *Then there is a complex number* c *such that* $|c| = 1$ *and* $cT \in W_1^1$. *Moreover,* T *is a linear (resp.* C^**) extreme point of* W_1 *if and only if whenever* T_1, T_2 *are matrices in* W_1^1 *such that* cT *is a linear (resp.* C^**) convex combination of* T_1 *and* T_2*, then* T_1 *and* T_2 *equal* cT *(resp.* T_1 *and* T_2 *are unitarily equivalent to* cT*).*

PROOF. We prove the corollary in the C^* case; the linear case follows similarly. If $w(T) = 1$ then there is a complex number λ in $W(T)$ such that $|\lambda| = 1$; clearly $1 \in W(\bar{\lambda}T)$ and we can choose $c = \bar{\lambda}$. Now T is a C^*-convex combination of T_3 and T_4 in W_1 if and only if cT is the same combination of cT_3 and cT_4, and Corollary 2.4 guarantees that cT_3 and cT_4 are in W_1^1. ■

In view of Corollary 2.5 we will say that T is linearly or C^*-extreme in W_1^1 if T lies in W_1^1 and satisfies the condition for cT in the corollary.

THEOREM 2.6. *Suppose that* $1 \in W(T)$. *Let* f *and* g *be unit vectors such that* $(Tf, f) = 1$ *and* g *is orthogonal to* f. *Let* $\operatorname{tr} T$ *and* $\det T$ *represent the trace and determinant of* T *respectively. The following statements are equivalent:*

(a) $w(T) = 1$;

(b) $(Tf, g) = -(f, Tg)$ *and* $\operatorname{tr} T - \det T$ *is real and greater than or equal to* $|\det T|$.

PROOF. We use Lemma 2.2. In the basis $\{f, g\}$ we can write $T = \begin{pmatrix} 1 & \alpha \\ \beta & \gamma \end{pmatrix}$. A matrix computation yields

$$\operatorname{Re}(I - zT) = \begin{pmatrix} \operatorname{Re}(1 - z) & (-z\alpha - \bar{z}\bar{\beta})/2 \\ (-z\beta - \bar{z}\bar{\alpha})/2 & \operatorname{Re}(1 - z\gamma) \end{pmatrix}.$$

For $|z| = 1$, $\operatorname{Re}(1 - z) \geq 0$ and by an elementary fact from linear algebra the selfadjoint matrix above is positive semidefinite if and only if its determinant is nonnegative. We have

$$\det \operatorname{Re}(I - zT) = \operatorname{Re}(1 - z)\operatorname{Re}(1 - z\gamma) - \tfrac{1}{4}|z\alpha + \bar{z}\bar{\beta}|^2.$$

By Lemma 2.2, T is in W_1 if and only if the right-hand side of the last equation (which we denote by $R(z)$) is nonnegative for all z such that $|z| = 1$.

First, suppose that $T \in W_1$. Then $R(1) \geq 0$ and it follows that $|\alpha + \bar{\beta}| = 0$, that is, $\beta = -\bar{\alpha}$, which establishes the condition $(Tf, g) = -(f, Tg)$. Using this fact we can rewrite $R(z)$ as

$$\begin{aligned} R(z) &= \operatorname{Re}(1 - z)\operatorname{Re}(1 - z\gamma) - |\alpha|^2(\operatorname{Im} z)^2 \\ &= \operatorname{Re}(1 - z)\left[\operatorname{Re}(1 - z\gamma) - |\alpha|^2\operatorname{Re}(1 + z)\right]. \end{aligned}$$

Since $\operatorname{Re}(1 - z) > 0$ whenever $|z| = 1$ and $z \neq 1$, the condition $R(z) \geq 0$ becomes, for $z \neq 1$,

$$\operatorname{Re}(1 - z\gamma) - |\alpha|^2 \operatorname{Re}(1 + z) \geq 0.$$

In fact this inequality persists for $z = 1$ as well, by the continuity of the left-hand side. We now have

$$\operatorname{Re}\big(z(\gamma + |\alpha|^2)\big) \leqslant 1 - |\alpha|^2, \quad \text{for } |z| = 1. \tag{1}$$

It is clear that the maximum value of $\operatorname{Re}(z(\gamma + |\alpha|^2))$ is $|\gamma + |\alpha|^2|$, attained either for all z (if $\gamma = -|\alpha|^2$) or when $z = (\bar{\gamma} + |\alpha|^2)/|\gamma + |\alpha|^2|$. Thus the requirement is that

$$|\gamma + |\alpha|^2| \leqslant 1 - |\alpha|^2, \tag{2}$$

which is the same as $|\det T| \leqslant \operatorname{tr} T - \det T$. The fact that this condition, together with the equation $\beta = -\bar{\alpha}$, yields the original hypothesis $w(T) = 1$, follows by a reversal of the above argument. ∎

THEOREM 2.7. *Let* $T \in W_1^1$.

(i) *If* $\operatorname{tr} T \neq 0$ *then* T *is linearly extreme in* W_1^1 *if and only if* $\operatorname{tr} T - \det T = |\det T|$.

(ii) *If* $\operatorname{tr} T = 0$ *then* T *is linearly extreme in* W_1^1 *if and only if* $\det T = 0$, *that is,* T *is nilpotent.*

PROOF. As in Theorem 2.6 we represent T by the matrix $\left(\begin{smallmatrix} 1 & \alpha \\ -\bar{\alpha} & \gamma \end{smallmatrix}\right)$, where $|\gamma + |\alpha|^2| \leqslant 1 - |\alpha|^2$. Observe that if $T = \lambda T_1 + (1 - \lambda) T_2$ for $T_1, T_2 \in W_1^1$ and $0 < \lambda < 1$, we can assume without harm that $\lambda = \frac{1}{2}$. Thus T fails to be linearly extreme if and only if there exists a nonzero matrix J such that $T + J$ and $T - J$ are in W_1^1. Since $|1 + \varepsilon| \leqslant 1$ and $|1 - \varepsilon| \leqslant 1$ imply $\varepsilon = 0$, T_1 and T_2 have the same form as T and J can be represented as

$$J = \begin{pmatrix} 0 & \delta \\ -\bar{\delta} & \eta \end{pmatrix}.$$

(i) Suppose $\operatorname{tr} T \neq 0$, that is, $\gamma \neq -1$. If $\operatorname{tr} T - \det T > |\det T|$, choose $\delta = 0$ and η sufficiently small that $|\gamma + \eta + |\alpha|^2| \leqslant 1 - |\alpha|^2$ and $|\gamma - \eta + |\alpha|^2| \leqslant 1 - |\alpha|^2$. Then $T + J$ and $T - J$ are in W_1^1 and T is not extreme.

Now let $\operatorname{tr} T - \det T = |\det T|$, that is, $|\gamma + |\alpha|^2| = 1 - |\alpha|^2$, and suppose that J has the above form with $T + J$ and $T - J$ in W_1^1. We must show that $\delta = \eta = 0$. If $\gamma + |\alpha|^2 = 0$, then also $1 - |\alpha|^2 = 0$ and it follows that $\gamma = -1$, contrary to assumption; thus $\gamma + |\alpha|^2 \neq 0$. Let $z_0 = (\bar{\gamma} + |\alpha|^2)/|\gamma + |\alpha|^2|$; clearly $|z_0| = 1$ and

$$\operatorname{Re}\big(z_0(\gamma + |\alpha|^2)\big) = 1 - |\alpha|^2. \tag{3}$$

By the proof of Theorem 2.6 (inequality (1)) and the fact that $T \pm J \in W_1^1$, we know that

$$\operatorname{Re}\big(z_0(\gamma + \eta + |\alpha + \delta|^2)\big) \leqslant 1 - |\alpha + \delta|^2 \tag{4}$$

and

$$\operatorname{Re}\big(z_0(\gamma - \eta + |\alpha - \delta|^2)\big) \leqslant 1 - |\alpha - \delta|^2. \tag{5}$$

Adding,

$$2 \operatorname{Re}\big(z_0(\gamma + |\alpha|^2 + |\delta|^2)\big) \leqslant 2(1 - |\alpha|^2 - |\delta|^2),$$

and using (3), $2|\delta|^2 \operatorname{Re} z_0 \leqslant -2|\delta|^2$, or $2|\delta|^2(1 + \operatorname{Re} z_0) \leqslant 0$.

If $z_0 = -1$ then $\gamma + |\alpha|^2$ is real and negative and the fact that $|\gamma + |\alpha|^2| = 1 - |\alpha|^2$ implies $\gamma = -1$, which is forbidden. Thus $\delta = 0$ and condition (2), applied to T, $T + J$, and $T - J$, gives

$$|\gamma + |\alpha|^2| = 1 - |\alpha|^2, \qquad |\gamma + \eta + |\alpha|^2| \leq 1 - |\alpha|^2$$

and $|\gamma - \eta + |\alpha|^2| \leq 1 - |\alpha|^2$.

Hence $\eta = 0$ and T is extreme.

(ii) Next suppose tr $T = 0$, that is, $\gamma = -1$. In this case the condition $|\gamma + |\alpha|^2| \leq 1 - |\alpha|^2$ becomes simply the requirement that $|\alpha| \leq 1$. If $|\alpha| < 1$ then for sufficiently small real δ, $|\alpha \pm \delta| < 1$ and T can be represented as

$$\begin{pmatrix} 1 & \alpha \\ -\bar{\alpha} & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \alpha + \delta \\ -\bar{\alpha} - \delta & -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & \alpha - \delta \\ -\bar{\alpha} + \delta & -1 \end{pmatrix}.$$

On the other hand, if $|\alpha| = 1$ there is no nonzero complex δ for which $|\alpha \pm \delta| \leq 1$, and T is extreme. To see that T is nilpotent under the conditions $\gamma = -1$ and $|\alpha| = 1$, observe that det $T = 0$ and tr $T = 0$ and consider the upper triangular form of T. ■

COROLLARY 2.8. *If T is linearly extreme in W_1 then the eigenvalues of T have equal absolute value.*

PROOF. If $w(T) < 1$ then for sufficiently small $\varepsilon > 0$, $w(T + \varepsilon) < 1$ and $w(T - \varepsilon) < 1$. Since $T = \frac{1}{2}(T + \varepsilon) + \frac{1}{2}(T - \varepsilon)$, T would not be extreme in this case. Thus we assume that $w(T) = 1$.

The theorem is the same if T is replaced by cT, where $|c| = 1$, so we can suppose that $T \in W_1^1$. Let $re^{i\theta}$ and $se^{i\phi}$ be the eigenvalues of T and note that $r, s \leq 1$. If tr $T = re^{i\theta} + se^{i\phi} = 0$ then $re^{i\theta} = -se^{i\phi}$ and there is nothing to prove. Suppose then that tr $T \neq 0$, so that case (i) of the theorem applies, and we have

$$re^{i\theta} + se^{i\phi} = rse^{i(\theta+\phi)} + rs.$$

Multiplying both sides of this equation by $e^{-i(\theta+\phi)/2}$, we obtain

$$\begin{aligned} re^{i(\theta-\varphi)/2} + se^{i(\phi-\theta)/2} &= rse^{i(\theta+\phi)/2} + rse^{-i(\theta+\phi)/2} \\ &= 2rs\cos(\theta+\phi)/2. \end{aligned}$$

Thus the imaginary part of the left-hand side is 0, that is,

$$r\sin(\theta - \phi)/2 + s\sin(\phi - \theta)/2 = 0, \quad \text{or} \quad (r - s)\sin(\theta - \phi)/2 = 0.$$

If $\sin(\theta - \phi)/2 \neq 0$ then $r = s$ immediately. If $\sin(\theta - \phi)/2 = 0$ then $\theta = \phi$ and the original equation reduces to

$$r + s = 2rs\cos\theta \quad \text{or} \quad \frac{1}{r} + \frac{1}{s} = 2\cos\theta.$$

Since $1/r + 1/s \geq 2$ and $2\cos\theta \leq 2$, it must be that $\theta = 0$ and $r = s = 1$. Thus in all cases $r = s$. ■

We can now obtain a geometric characterization of the linear extreme points of W_1. Recall that $W(T)$ is an elliptical disk.

THEOREM 2.9. *Let $w(T) \leq 1$. T is linearly extreme in W_1 if and only if one of the following conditions holds*:

(I) *$W(T)$ is the entire unit disk*;

(II) *$W(T)$ consists of precisely one point, which lies on the unit circle*;

(III) *$W(T)$ intersects the unit circle in exactly two nonantipodal points*;

(IV) *$W(T)$ intersects the unit circle in precisely one point, which lies on the minor axis of $W(T)$, and the unit circle is the osculating circle of $W(T)$ at that point* (*that is, the curvatures are equal*).

PROOF. Conditions (I)–(IV) require that $W(T)$ intersect the unit circle, as does linear extremity of T. Furthermore, (I)–(IV) are unchanged if T is multiplied by a complex scalar of modulus one. Thus we can assume that $w(T) = 1$ and that $T \in W_1^1$. Let T be represented in terms of α and γ as in Theorem 2.7 and let $R(z)$ be as defined in Theorem 2.6. Lemma 2.2 shows that if $|z| = 1$, then $z \in W(T)$ if and only if $R(\bar{z}) = 0$.

(i) Suppose first that $\operatorname{tr} T = 0$. By Theorem 2.7, T is linearly extreme if and only if it is nilpotent. Lemma 2.1 easily shows that when $w(T) = 1$, T is nilpotent exactly when $W(T)$ is the unit disk, that is, condition (I) holds.

Next suppose that $\operatorname{tr} T \neq 0$. We assert that if $\det T = 0$, then neither is T linearly extreme, nor can it satisfy any of conditions (I) – (IV). It follows easily from (i) of Theorem 2.7 that T is not extreme. If $\det T = 0$ then one of the eigenvalues of T must be 0, and by the trace condition the other is nonzero, say λ. Thus the foci of $W(T)$ are 0 and λ, and it is easy to see that such an ellipse cannot satisfy any of (I)–(IV).

Thus we assume that $\operatorname{tr} T$ and $\det T$ are both nonzero. Let

$$z_0 = (\gamma + |\alpha|^2)/|\gamma + |\alpha|^2| = \det T/|\det T|,$$

and let the eigenvalues of T be λ and μ. It is shown in Theorem 2.6 that if $R(\bar{z}) = 0$ for any value of z other than $z = 1$, then $R(\bar{z}_0) = 0$. We consider three cases which clearly exhaust the remaining possibilities.

(ii) $\operatorname{tr} T \neq 0$, $\lambda = \mu$. Condition (i) of Theorem 2.7 becomes $2\lambda - \lambda^2 = |\lambda|^2$, and it is easy to check that this equation has only two solutions in the unit disk, namely $\lambda = 0$ and $\lambda = 1$, the first of which is ruled out by the trace condition. But if $\lambda = 1$ is to be the focus of an ellipse lying inside the unit circle, it must be that the ellipse is the single point 1. Thus in this case extremity is equivalent to condition (II).

(iii) $\operatorname{tr} T \neq 0$, $\lambda \neq \mu$, $z_0 \neq 1$. It is shown in Theorems 2.6 and 2.7 that if T is extreme, $R(\bar{z}_0) = 0$ and $W(T)$ intersects the unit circle at 1 and at z_0. The argument of Theorem 2.7 shows that the extremity of T and the fact that $\operatorname{tr} T \neq 0$ imply that $z_0 \neq -1$ and thus that condition (III) holds. On the other hand, if (III) holds then (as in Theorem 2.6) z_0 must be a point of intersection of $W(T)$ and the unit circle, whence $R(\bar{z}_0) = 0$ and thus $|\gamma + |\alpha|^2| = 1 - |\alpha|^2$, that is, $\operatorname{tr} T - \det T = |\det T|$ and T is extreme.

(iv) $\operatorname{tr} T \neq 0$, $\lambda \neq \mu$, $z_0 = 1$. In this case $W(T)$ intersects the unit circle in exactly one point. Suppose first that T is extreme. Consider the perpendicular bisector L of

λ and μ. The minor axis of $W(T)$ lies along L, and since $|\lambda| = |\mu|$ by Corollary 2.8, L passes through the origin. Thus both $W(T)$ and the unit circle are symmetric about L and it follows that the single intersection point of the two sets must lie on L, that is, at an endpoint of the minor axis. Since the intersection point is 1, L is the real axis and $\mu = \bar{\lambda}$. If we represent λ as $x + iy$, the ellipse $W(T)$ has semiminor axis $a = (1 - x)$ and semimajor axis $b = ((1 - x)^2 + y^2)^{1/2}$, and the equation tr $T -$ det $T = |\det T|$ becomes $x - x^2 = y^2$, so that $b = \sqrt{a}$. It is a standard fact that an ellipse with semiminor and semimajor axes a and b respectively has curvature b^2/a at each end of the minor axis. Thus the curvature of $W(T)$ at 1 is 1 and the unit circle is the osculating circle, so condition (IV) holds.

Finally, suppose condition (IV) holds. Then $\mu = \bar{\lambda}$ and the curvature condition implies that $x - x^2 = y^2$, that is, $\operatorname{Re} \lambda = |\lambda|^2$. From these facts it follows readily that tr $T -$ det $T = |\det T|$, and we are done. ■

The above theorem has the consequence that the linear extremity of T is determined by $W(T)$ alone, not by any other properties of T. As we have observed, the C^*-extreme points of W_1 lie among the linearly extreme points. We show below that the matrices in classes (I) and (II) are C^*-extreme (Theorem 2.10), whereas no matrix in class (III) or (IV) is C^*-extreme (Theorems 2.16 and 2.17). As in the case of linear extremity, C^*-extremity is most easily discussed for the set W_1^1. To be specific, we say that T is C^*-extreme in W_1^1 if T lies in W_1^1 and if, whenever P_1 and P_2 are invertible positive matrices with $P_1^2 + P_2^2 = I$ and for which there exists matrices T_1 and T_2 in W_1^1 with $T = P_1T_1P_1 + P_2T_2P_2$, then T_1 and T_2 must be unitarily equivalent to T.

THEOREM 2.10. *The identity matrix and all nilpotent matrices in W_1^1 are C^*-extreme in W_1^1.*

PROOF. First, suppose that $I = P_1T_1P_1 + P_2T_2P_2$, where $P_1^2 + P_2^2 = I$ and P_1 and P_2 are positive and invertible. By Lemma 2.3, for any unit vector f we have that the complex number 1 is a convex combination of (T_1f_1, f_1) and (T_2f_2, f_2), where $f_i = P_if/\|P_if\|$. Since $|(T_if_i, f_i)| \leq 1$, necessarily $(T_if_i, f_i) = 1$, and because f was arbitrary it follows that $(T_iu, u) = 1$ for $i = 1, 2$ and for any unit vector u. From this fact it is easy to see that $T_1 = T_2 = I$.

Next, suppose that T is nilpotent and is a proper C^*-convex combination of T_1 and T_2. By Lemma 2.1, $W(T)$ consists of the entire unit disc, and by Corollary 2.4 so do $W(T_1)$ and $W(T_2)$. A second application of Lemma 2.1 shows that the eigenvalues of T_1 and T_2 are all 0 and thus that T_1 and T_2 are nilpotent. Thus T and T_1 can be represented (in perhaps different bases) as

$$T = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}.$$

Since $w(T) = w(T_1) = 1$ is necessary that $|\alpha| = |\beta| = 2$, and the unitary operator

$$U = \begin{pmatrix} \bar{\beta}/\bar{\alpha} & 0 \\ 0 & 1 \end{pmatrix}$$

implements the equivalence between T and T_1. The equivalence of T and T_2 follows similarly. ■

The next sequence of technical results culminates in Theorem 2.16, in which we show that the operators in class (III) of Theorem 2.9 are never C^*-extreme.

LEMMA 2.11. *Let T, S be nonnilpotent matrices in W_1 and let f and g be independent unit vectors for which*

$$(Sf, f) = (Tf, f) = e^{i\theta} \quad \text{and} \quad (Sg, g) = (Tg, g) = e^{i\psi},$$

where $e^{i\theta} \neq -e^{i\psi}$. Then $S = T$.

PROOF. Without loss of generality we assume that $e^{i\theta} = 1$; then $e^{i\psi} \neq -1$. In an appropriate basis, we can write

$$T = \begin{pmatrix} 1 & \alpha \\ -\bar{\alpha} & \gamma \end{pmatrix}, \qquad S = \begin{pmatrix} 1 & \alpha' \\ -\bar{\alpha}' & \gamma' \end{pmatrix}.$$

We may assume also that (f, g) is real and positive, so that $g = (\cos \beta, e^{i\phi} \sin \beta)$ for appropriate $\beta \in [0, \pi/2]$ and $\phi \in [0, 2\pi]$. It will suffice to show that the stated conditions determine α and γ uniquely. By the above remarks, T is linearly extreme in W_1^1 and thus

$$e^{i\psi} = \frac{\gamma + |\alpha|^2}{|\gamma + |\alpha|^2|} \tag{6}$$

and

$$|\gamma + |\alpha|^2| = 1 - |\alpha|^2. \tag{7}$$

Moreover,

$$\operatorname{Re}(I - e^{i\psi}T) = \begin{pmatrix} 1 - \cos\psi & i\alpha \sin\psi \\ -i\bar{\alpha}\sin\psi & 1 - t\cos(\omega - \psi) \end{pmatrix}$$

where $\gamma = te^{i\omega}$. The equation $\operatorname{Re}(I - e^{-i\psi}T)g = 0$ becomes

$$(1 - \cos\psi)\cos\beta + i\alpha e^{i\phi}\sin\psi\sin\beta = 0, \tag{8}$$

$$-i\bar{\alpha}\sin\psi\cos\beta + e^{i\phi}\sin\beta - te^{i\phi}\sin\beta\cos(\omega - \psi) = 0. \tag{9}$$

Since f and g are linearly independent, $\sin\beta \neq 0$. If $\sin\psi = 0$, then (since $e^{i\psi} \neq -1$), $\psi = 0$ and (9) becomes $\sin\beta[1 - t\cos\omega] = 0$ so that $t\cos\omega = 1$. The fact that $|\gamma| < 1$ means that $\gamma = 1$ and hence by (7), $\alpha = 0$. Thus in this case $T = I$.

If $\sin\psi \neq 0$ then α is determined from equation (8). Equations (6) and (7) determine the argument and modulus of $\gamma + |\alpha|^2$ and thus γ is determined. ■

COROLLARY 2.12. *Let $T \in W_1$ and let f, f' be unit vectors such that $(Tf, f) = (Tf', f')$ and $|(Tf, f)| = 1$. Then either f and f' are linearly dependent or T is a multiple of the identity.*

PROOF. If T is nilpotent, it is unitarily equivalent to $\left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right)$ and it is easy to check that the conclusion holds for the latter matrix. If T is not nilpotent, let $(Tf, f) = e^{i\theta}$, let $S = e^{i\theta}I$ and use the lemma. ■

COROLLARY 2.13. *Let T, S be nonnilpotent operators in W_1. Suppose $e^{i\phi} \neq \pm e^{i\psi}$ and that $e^{i\phi}$ and $e^{i\psi}$ both lie in $W(T) \cap W(S)$. Let $\{f, g\}$ and $\{f', g'\}$ be pairs of independent unit vectors such that $(Tf, f) = (Sf', f') = e^{i\phi}$ and $(Tg, g) = (Sg', g') = e^{i\psi}$. Then T and S are unitarily equivalent if and only if $|(f, g)| = |(f', g')|$.*

PROOF. Since $e^{i\phi} \neq e^{i\psi}$, T and S are not multiples of the identity. Suppose that U is unitary and that $S = U^*TU$. Then

$$(TUf', Uf') = (U^*TUf', f') = (Sf', f') = (Tf, f)$$

and Corollary 2.12 ensures that the set $\{f, Uf'\}$ is linearly dependent; say $Uf' = e^{i\theta}f$. Similarly $Ug' = e^{i\mu}g$ and $|(f, g)| = |(e^{i\theta}f, e^{i\mu}g)| = |(Uf', Ug')| = |(f', g')|$.

Now suppose $|(f, g)| = |(f', g')|$. By multiplying g and g' by complex numbers of modulus one we can assume that (f, g) and (f', g') are real and positive. Let $Uf' = f$ and $Ug' = g$. Then

$$\|U(\alpha f' + \beta g')\|^2 = \|\alpha f + \beta g\|^2 = |\alpha|^2 + |\beta|^2 + 2 \operatorname{Re} \alpha\bar{\beta}(f, g)$$
$$= |\alpha|^2 + |\beta|^2 + 2 \operatorname{Re} \alpha\bar{\beta}(f', g') = \|\alpha f' + \beta g'\|^2.$$

Hence U is unitary. Let $S_{i0} = U^*TU$; then $(S_0 f', f') = (TUf', Uf') = (Tf, f) = e^{i\phi}$ and likewise $(S_0 g', g') = e^{i\psi}$. By the lemma we conclude that $S_0 = S$ and the proof is complete. ■

THEOREM 2.14. *Let $T \in W_1^1$ and suppose that $e^{i\psi} \in W(T)$, where $e^{i\psi} \neq \pm 1$. Let f and g be unit vectors for which $(Tf, f) = 1$ and $(Tg, g) = e^{i\psi}$. Then*

$$|(f, g)| < |\cos(\psi/2)|.$$

Conversely, if f and g are linearly independent unit vectors satisfying the last inequality then there exists a $T \in W_1^1$ for which $(Tf, f) = 1$ and $(Tg, g) = e^{i\psi}$, and T is unique.

PROOF. We write T in an orthonormal basis $\{f, h\}$ as

$$T = \begin{pmatrix} 1 & \alpha \\ -\bar{\alpha} & \gamma \end{pmatrix}.$$

Since $e^{i\psi} \neq 1$, T is extreme in W_1^1.

Without loss of generality, we suppose that (f, g) is real and positive so that g may be written as $(\cos\beta, e^{i\phi}\sin\beta)$ for $\beta \in [0, \pi/2]$ and $\phi \in [0, 2\pi]$. As before, the condition $\operatorname{Re}(I - e^{-i\psi}T)g = 0$ yields

$$(1 - \cos\psi)\cos\beta + i\alpha e^{i\phi}\sin\psi\sin\beta = 0.$$

Since $e^{i\psi} \neq \pm 1$, $\sin\psi \neq 0$; the hypotheses guarantee that f and g are linearly independent, so $\sin\beta \neq 0$ as well. Furthermore if $\cos\beta = 0$ then $|(f, g)| = 0$ and there is nothing to prove, so we suppose that $\cos\beta \neq 0$.

We have $(1 - \cos\psi)\cos\beta = -i\alpha e^{i\phi}\sin\psi\sin\beta$, and since $|\alpha| < 1$,

$$|(1 - \cos\psi)\cos\beta| < |\sin\psi\sin\beta|.$$

Thus

$$\tan^2\beta > \left(\frac{1 - \cos\psi}{\sin\psi}\right)^2 \tag{10}$$

and

$$\sec^2\beta = 1 + \tan^2\beta > 1 + \left(\frac{1-\cos\psi}{\sin\psi}\right)^2 = \frac{2}{1+\cos\psi},$$

which leads to

$$\cos^2\beta < \frac{1}{2}(1+\cos\psi) = \cos^2\frac{\psi}{2}$$

and the result follows.

To prove the converse we assume that f, g and ψ are given, with $|(f, g)| < |\cos\psi/2|$; as before we suppose that $(f, g) = \cos\beta$ is real and positive. Since $e^{i\psi} \neq \pm 1$, $|\cos\psi/2| \neq 1$ and thus $0 < \cos\beta < 1$. Choose a vector h such that $h \perp f$ and $(g, h) = i\sin\beta$. Inequality (10) persists in this case and we define

$$\alpha = \cot\beta\frac{1-\cos\psi}{\sin\psi} \quad \text{and} \quad \gamma = e^{i\psi} - |\alpha|^2(1+e^{i\psi}).$$

Let $T = \left(\begin{smallmatrix}1 & \alpha\\ -\bar{\alpha} & \gamma\end{smallmatrix}\right)$, written in the basis $\{f, h\}$. We must show that $w(T) = 1$ and that $(Tg, g) = e^{i\psi}$.

Since $|\gamma + |\alpha|^2| = |e^{i\psi} - |\alpha|^2e^{i\psi}| = 1 - |\alpha|^2$ we have $w(T) = 1$. To show that $(Tg, g) = e^{i\psi}$ it is sufficient to show that $\operatorname{Re}(I - e^{-i\psi}T)g = 0$, that is,

$$\begin{pmatrix} 1-\cos\psi & i\alpha\sin\psi \\ -i\alpha\sin\psi & \operatorname{Re}(1-e^{-i\psi}\gamma)\end{pmatrix}\begin{pmatrix}\cos\beta \\ i\sin\beta\end{pmatrix} = 0,$$

which is equivalent to the two equations

$$(1-\cos\psi)\cos\beta - \alpha\sin\psi\sin\beta = 0, \tag{11}$$

$$-i\alpha\sin\psi\cos\beta + i\sin\beta\operatorname{Re}(1-e^{-i\psi}\gamma) = 0. \tag{12}$$

By definition of α,

$$\begin{aligned}(1-\cos\psi)&\cos\beta - \alpha\sin\psi\sin\beta \\ &= (1-\cos\psi)\cos\beta - \cos\beta\frac{1-\cos\psi}{\sin\psi}\sin\psi = 0,\end{aligned}$$

so equation (11) holds. We also have

$$\begin{aligned}-i\alpha\sin\psi&\cos\beta + i\sin\beta\operatorname{Re}(1-e^{-i\psi}\gamma)\\ &= -i\alpha\sin\psi\cos\beta + i\sin\beta\operatorname{Re}(|\alpha|^2(1+e^{-i\psi}))\\ &= i\alpha[-\sin\psi\cos\beta + \alpha\sin\beta(1+\cos\psi)]\\ &= i\alpha\left[-\sin\psi\cos\beta + \cos\beta\frac{1-\cos\psi}{\sin\psi}(1+\cos\psi)\right] = 0.\end{aligned}$$

The fact that T is unique follows from Lemma 2.11. The proof is complete. ■

LEMMA 2.15. *Let f and g be linearly independent unit vectors and let K be a positive number such that $|(f, g)| < K < 1$. Then there exist positive invertible matrices P_1 and P_2 such that $P_1^2 + P_2^2 = I$,*

$$|(P_if, P_ig)| < K\|P_if\|\,\|P_ig\|, \quad \textit{and} \quad |(P_if, P_ig)| \neq |(f, g)|\,\|P_if\|\,\|P_ig\|,$$

for $i = 1, 2$.

PROOF. The conditions can be written in the following form:

$$K^2(P_i^2f,f)(P_i^2g,g) - |(P_i^2f,g)|^2 > 0, \tag{13}$$

$$|(f,g)|^2(P_i^2f,f)(P_i^2g,g) - |(P_i^2f,g)|^2 \neq 0. \tag{14}$$

Let Q be a positive matrix to be chosen later. We identify P_1^2 with $\frac{1}{2} + \delta Q$ and P_2^2 with $\frac{1}{2} - \delta Q$, for some small positive δ; our task is to find an appropriate Q and δ. Let H and G be real-valued functions of the real variable t defined as follows:

$$H(t) = K^2\left(\left(\tfrac{1}{2} + tQ\right)f,f\right)\left(\left(\tfrac{1}{2} + tQ\right)g,g\right) - \left|\left(\left(\tfrac{1}{2} + tQ\right)f,g\right)\right|^2;$$

$$G(t) = |(f,g)|^2\left(\left(\tfrac{1}{2} + tQ\right)f,f\right)\left(\left(\tfrac{1}{2} + tQ\right)g,g\right) - \left|\left(\left(\tfrac{1}{2} + tQ\right)f,g\right)\right|^2.$$

The lemma will be proved provided we can find δ such that $H(\delta) > 0$, $H(-\delta) > 0$, $G(\delta) \neq 0$, and $G(-\delta) \neq 0$. Now H is a continuous function of t and

$$H(0) = \tfrac{1}{4}K^2 - \tfrac{1}{4}|(f,g)|^2 > 0.$$

Thus there is a positive number t_0 such that $H(t) > 0$ whenever $|t| < t_0$. Furthermore, $G(t)$ is a quadratic polynomial in the real variable t, and $G(0) = 0$; thus, unless $G(t)$ is identically zero, there is a number t_1 such that $G(t) \neq 0$ whenever $0 < |t| < t_1$, and by choosing any nonzero δ smaller than t_0 and t_1 such that $\frac{1}{2} \pm \delta Q$ are positive, our problem is solved. Thus we must show that for proper choice of Q the function $G(t)$ is nonconstant. Routine calculation yields the equations

$$G'(0) = \tfrac{1}{2}|(f,g)|^2\left[(Qf,f) + (Qg,g)\right] - \operatorname{Re}\left[(f,g)\left(\overline{Qf,g}\right)\right]$$

and

$$G''(0) = 2|(f,g)|^2(Qf,f)(Qg,g) - 2|(Qf,g)|^2.$$

If either of $G'(0)$, $G''(0)$ is nonzero then G is nonconstant. If $(f,g) = 0$, choose positive Q such that $(Qf,g) \neq 0$, ensuring that $G''(0) \neq 0$. On the other hand, if $(f,g) \neq 0$, choose Q such that $(Qf,g) = 0$ (this can always be done). This choice makes $G'(0) \neq 0$ and we are done. ■

THEOREM 2.16. *Let $T \in W_1$ and suppose that $W(T)$ contains at least two distinct complex numbers of absolute value one. Then T is C^*-extreme if and only if it is nilpotent, that is, if $W(T)$ contains the entire unit circle.*

PROOF. Multiplying T by a complex number of modulus one changes neither the hypothesis nor the conclusion, so we suppose that $T \in W_1^1$ and that there is a number $z_0 \in W(T)$ such that $|z_0| = 1$ and $z_0 \neq 1$. Represent T as

$$T = \begin{pmatrix} 1 & \alpha \\ -\bar{\alpha} & \gamma \end{pmatrix}$$

in an appropriate basis, and suppose first that $z_0 = -1$. Then $\det \operatorname{Re}(I + T) = 0$ and we have, as in the proof of Theorem 2.6, that $\operatorname{Re}(-(\gamma + |\alpha|^2)) = 1 - |\alpha|^2$. The fact that $T \in W_1^1$ means that $|\gamma + |\alpha|^2| \leq 1 - |\alpha|^2$, and it follows that $\gamma + |\alpha|^2$ is real and negative, so that $\gamma + |\alpha|^2 = |\alpha|^2 - 1$, that is, $\gamma = -1$. But now $\operatorname{tr} T = 0$, and by Theorem 2.7, T is linearly extreme if and only if it is nilpotent; the same is true for C^*-extremity by Theorem 2.10.

Next we suppose that $z_0 \neq \pm 1$. We already know that nilpotent matrices are C^*-extreme so we assume T is not nilpotent. Let f and g be linearly independent unit vectors for which $(Tf, f) = 1$ and $(Tg, g) = z_0$. If $z_0 = e^{i\psi}$ then $|(f, g)| < |\cos \psi/2|$ by Theorem 2.14. We assert that if $|(f, g)| = |\cos \psi/2|$ then T is nilpotent. For we can suppose as before that (f, g) is real and of appropriate sign so that $(f, g) = \cos \psi/2$ and that a vector h has been chosen orthogonal to f such that $(g, h) = i \sin \psi/2$. Consider the matrix $T_0 = \left(\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right)$. It is easy to check that $(T_0 f, f) = 1$, $(T_0 g, g) = e^{i\psi}$, and $T_0 \in W_1$ (by Theorem 2.6). Thus $T = T_0$ because of Theorem 2.14, but T_0 is nilpotent. Thus we reject the possibility that $|(f, g)| = |\cos \psi/2|$.

The rest follows easily from the preceding sequence of results. We know that $|(f, g)| < |\cos \psi/2|$, so by Lemma 2.15 there exist positive invertible matrices P_1 and P_2 as described there. For convenience we denote by f_1, g_1, f_2, g_2 the unit vectors $P_1 f/\|P_1 f\|$, $P_1 g/\|P_1 g\|$, $P_2 f/\|P_2 f\|$ and $P_2 g/\|P_2 g\|$, respectively.

The conditions in the conclusion of Lemma 2.15 can be written as $|(f_i, g_i)| < |\cos \psi/2|$ and $|(f_i, g_i)| \neq |(f, g)|$, for $i = 1, 2$. By Theorem 2.14 there exist matrices T_1, T_2 in W_1^1 such that

$$(T_1 f_1, f_1) = (T_2 f_2, f_2) = 1 \quad \text{and} \quad (T_1 g_1, g_1) = (T_2 g_2, g_2) = e^{i\psi} = z_0.$$

Now consider the matrix $T_0 = P_1 T_1 P_1 + P_2 T_2 P_2$. Since W_1 is a C^*-convex set, $T_0 \in W_1$. Moreover,

$$\begin{aligned}(T_0 f, f) &= (P_1 T_1 P_1 f, f) + (P_2 T_2 P_2 f, f) \\ &= (T_1 P_1 f, P_1 f) + (T_2 P_2 f, P_2 f) \\ &= \|P_1 f\|^2 (T_1 f_1, f_1) + \|P_2 f\|^2 (T_2 f_2, f_2) \\ &= \|P_1 f\|^2 + \|P_2 f\|^2 = 1,\end{aligned}$$

because $P_1^2 + P_2^2 = I$. Likewise, $(T_0 g, g) = e^{i\psi}$. Hence, by the uniqueness part of Theorem 2.14, $T_0 = T$. Observe next that the same sort of reasoning as before allows us to conclude that the condition $|(f_i, g_i)| \neq |\cos \psi/2|$ implies that T_i is not nilpotent, $i = 1, 2$. Finally, by Corollary 2.13, since $|(f_i, g_i)| \neq |(f, g)|$ we know that T_1 and T_2 are not unitarily equivalent to T. Thus T is a proper C^*-convex combination of T_1 and T_2 and T is not C^*-extreme. The proof is complete. ■

THEOREM 2.17. *Let T be a nonscalar linear extreme point of W_1 and suppose that $W(T)$ intersects the unit circle in precisely one point, which lies on the minor axis of $W(T)$, and that the unit circle is the osculating circle of $W(T)$ at that point. Then T is not C^*-extreme in W_1.*

PROOF. Multiplying T by a complex number of modulus one changes neither the hypotheses nor the conclusion, so we suppose that $T \in W_1^1$. Again, we choose an appropriate basis and represent T as $T = \left(\begin{smallmatrix} 1 & \alpha \\ -\bar{\alpha} & \gamma \end{smallmatrix}\right)$. Furthermore, conjugating T by a diagonal unitary, we see that we may, and do, assume that $\alpha = \bar{\alpha} > 0$.

As in Theorem 2.9, the hypotheses together with the fact that $1 \in W(T)$ yield that the minor axis of $W(T)$ lies along the real axis, so the eigenvalues of T are complex conjugates. Let $\lambda = a + bi$, $b > 0$, be one of the eigenvalues. Since $W(T)$ is contained in the unit disc, intersects it at only the point 1, and T is not scalar, we must have $0 < a < 1$.

By Theorem 2.7(i), we must have $\operatorname{tr} T - \det T = |\det T|$, which becomes $2a - (a^2 + b^2) = a^2 + b^2$, so that $b = \sqrt{a - a^2}$. To determine the entries of T in terms of a, we observe that

$$1 + \gamma = \operatorname{tr} T = 2a, \qquad \gamma + \alpha^2 = \det T = a^2 + b^2 = a,$$

so that $\gamma = 2a - 1$, $\alpha = \sqrt{1 - a}$.

Thus, T becomes

$$T = \begin{pmatrix} 1 & \sqrt{1-a} \\ -\sqrt{1-a} & 2a - 1 \end{pmatrix}.$$

Now to see that T is not C^*-extreme, fix c such that $a < c < 1$ and set

$$d_1 = \sqrt{\frac{a(1-c)}{c(1-a)}}, \qquad d_2 = \sqrt{\frac{a}{c}}.$$

Note that $0 < d_i < 1$ so that

$$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}\begin{pmatrix} 1 & \sqrt{1-c} \\ -\sqrt{1-c} & 2c-1 \end{pmatrix}\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$
$$+ \begin{bmatrix} \sqrt{1-d_1^2} & 0 \\ 0 & \sqrt{1-d_2^2} \end{bmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{bmatrix} \sqrt{1-d_1^2} & 0 \\ 0 & \sqrt{1-d_2^2} \end{bmatrix}$$
$$= \begin{pmatrix} 1 & \sqrt{1-a} \\ -\sqrt{1-a} & 2a-1 \end{pmatrix} = T$$

is a representation of T as a proper C^*-convex combination of terms which are not unitarily equivalent to T.

3. Unit operator interval.

THEOREM 3.1. *Assume that $\mathcal{H}$ is infinite dimensional, let $\mathcal{K}$ denote the compact operators, and let $\mathcal{P} = \{T \in \mathcal{B}(\mathcal{H}) | 0 \leq T \leq I\}$. Then each of the sets $\mathcal{P}$, $\mathcal{P} \cap \mathcal{K}$ and $\{T \in \mathcal{P} | I - T \in \mathcal{K}\}$ is singly generated as a closed C^*-convex set. Indeed if P is a projection (unequal to 0 or I) then* MCL$(P) = \mathcal{P}$ *if P has inifnite rank and corank,* MCL$(P) = \mathcal{P} \cap \mathcal{K}$ *if P has finite rank, and* MCL$(P) = \{T \in \mathcal{P} | I - T \in \mathcal{K}\}$ *if P has finite corank.*

PROOF. First assume that P is a projection with infinite rank and corank. Any other projection with infinite rank and corank is unitarily equivalent to P, hence lies in MCL(P). In particular, $P^\perp \in$ MCL(P). Both 0 and I are C^*-convex combinations of P and $P^\perp$. (Namely $0 = P^\perp(P)P^\perp + P(P^\perp)P$ and $I = P(P)P + P^\perp(P^\perp)P^\perp$.) Therefore both 0 and I belong to MCL(P). If T is any element of $\mathcal{P}$ then, T is a C^*-convex combination of 0 and I. Indeed, $T = T^{1/2}(I)T^{1/2} + (I - T)^{1/2}(0)(I - T)^{1/2}$. Thus $\mathcal{P} \subseteq$ MCL(P); the reverse inclusion is obvious.

Next consider the case in which P is a nonzero projection with finite rank. Since $\mathcal{P} \cap \mathcal{K}$ is a closed C^*-convex set which contains P, $\text{MCL}(P) \subseteq \mathcal{P} \cap \mathcal{K}$. Any projection Q with the same rank as P is unitarily equivalent to P and so must also belong to $\text{MCL}(P)$. If Q has the same rank as P and also commutes with P, then the projection QP can be written as a C^*-convex combination of P and Q; viz., $QP = P^{\perp}(P)P^{\perp} + P(Q)P$. But Q can be chosen so that QP has any desired rank less than the rank of P (including rank 0); hence all projections whose rank is less than the rank of P lie in $\text{MCL}(P)$. Next observe that if Q_1 and Q_2 are any two mutually orthogonal projections in $\text{MCL}(P)$ then $Q_1 + Q_2 \in \text{MCL}(P)$; just write $Q_1 + Q_2 = Q_1(Q_1)Q_1 + Q_1^{\perp}(Q_2)Q_1^{\perp}$. It now follows that every finite rank projection is in $\text{MCL}(P)$. If T is an arbitrary finite rank operator in $\mathcal{P}$ and if Q is the range projection of T then 0 and Q lie in $\text{MCL}(P)$ and hence $T = T^{1/2}(Q)T^{1/2} + (I - T)^{1/2}(0)(I - T)^{1/2}$ also lies in $\text{MCL}(P)$. Since $\text{MCL}(P)$ is closed, it now follows that $\mathcal{P} \cap \mathcal{K} \subseteq \text{MCL}(P)$.

The final claim, that if P is a projection with finite nonzero corank then $\text{MCL}(P) = \{T \in \mathcal{P} \mid I - T \in \mathcal{K}\} = I - \mathcal{P} \cap \mathcal{K}$, follows from the paragraph above and the routine observation that, for any operator S, $\text{MCL}(I - S) = I - \text{MCL}(S)$. Apply this observation with $S = I - P$.

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